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14. ABSTRACT

Despite the great amount of research devoted to the visibility problem, very little attention has been devoted so far to measure or compare visibility data in a way which is consistent with "human-based" goals, as of interest in typical military applications. In fact, terrains are usually compared using mathematical norms which are unable to measure distances in terms of their visibility properties. The objective of this report is to introduce a functional-theoretic framework to deal with the notion of visibility and develop mathematically well-defined and

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A human-goal-based approach to the analysis and processing of visibility data

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A human-goal-based approach to the analysis and processing of visibility data

Demetrio Labate[*], Gitta Kutyniok[†]

July 30, 2010

Abstract

Despite the great amount of research devoted to the visibility problem, very little attention has been devoted so far to measure or compare visibility data in a way which is consistent with "human-based" goals as of interest in typical military applications. In fact, terrains are usually compared using mathematical norms which are unable to measure distances in terms of their visibility properties. The objective of this report is to introduce a functional-theoretic framework to deal with the notion of visibility and develop mathematically well-defined and practically effective tools to process visibility information, including the comparison, analysis and classification of visibility data. In particular, a new notion of *difference of visibility distance* is introduced, as a theoretical and computational tool for the comparison of terrain models from the point of their visibility properties.

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Contents

1 Introduction

The visibility problem is a among the most important research topics in computational geometry and is of interest in both military applications such as surveillance and reconnaissance and civilian applications such as robotics, geographic information systems and computer graphics. Due to its importance, this problem has been extensively investigated and several efficient algorithms have proposed to produce detailed terrain models, compute the visibility regions and address viewpoint-based placement problems such as the classical "museum problem" and its generalizations (see, for example, [1, 13, 17, 18]).

However, despite all this effort, certain very relevant aspects of the notion of visibility have not received the proper amount of attention. In particular, there is no satisfactory way to measure or compare visibility data in a way which is consistent with "human-based" goals. This is of particular importance to effectively assess, for example, how well a visibility model approximates the ground truth, or to compare different visibility models and algorithms. In addition, very little effort has been devoted in the scientific literature to examine the functional-analytic properties associated with the notion of visibility. Indeed, in virtually all studies, the visibility problem has been examined in the context of discrete rather than continuous data. While traditional methods from computer science and computational geometry are very effective to compute visibility structures and deal with many practical issues related to visibility computations, our point of view is that, by making use of the continuous setting, one can take advantage of the power of functional analysis to gain a deeper insight into the visibility problem and develop more sophisticated tools for the analysis and processing of visibility data. .

The main goal of this work is to introduce a novel mathematical approach to deal with the notion of visibility. By setting a proper functional-analytic framework, we will develop mathematically well-defined and practically effective tools to better process visibility information, including the comparison, analysis and classification of visibility data in a way which is consistent with "human-based" goals.

The report will be organized as follows. Section 1.1 contains the notation and definitions which will be used throughout the report. Section 2 sets the proper mathematical framework for the description and characterization of the viewshed function and sets the groundwork for next section. Section 3 contains the main results of this work. Specifically, a new definition of Difference of Visibility Distance is introduced and validated on a number of examples to show its consistency with human-based goals. Using this notion, continuity, equivalence and approximation properties on a simple model of univariate terrain models are examined.

1.1 Notation and Definitions

We start by introducing the basic definitions and notations which will be used throughout this report.

A *topographic surface* or *terrain* is defined as the graph of a bivariate function $f : Q \subset \mathbb{R}^2 \to \mathbb{R}$, that assigns a height or elevation $f(x)$ to every point x in a domain $Q \subset \mathbb{R}^2$. We will assume that $Q = [0,1]^2$ and that the function f is bounded.

In practical situation, the values of the function f are known only at a finite set $S \subset Q$ of sample points. This gives rise a discrete version of a terrain which are known as a *Digital Elevation Model* (DEM), consisting of elevation measures on the discrete set of points $S \subset Q$, where S is either a regular grid, or an irregularly scattered set of points. In the first case, we have a *Regular Square Grid* (RSG), which is obtained by partitioning S into equally sized rectangles. Using this model, the function f is defined piecewise over each rectangle, e.g., using a constant or bilinear function. Each rectangle can be further divided into two triangles, with linear interpolating functions defined on them. In the second case, when the points in S are irregular samples, we have a *Triangulate Irregular Network* (TIN), which is defined by introducing a triangulation T of the set S. Also in this case, f is piecewise defined, for example by using linear functions over each triangle of T. Hence in both cases the result is a polyhedral terrain, which is the graph of a piecewise linear function and provides an approximation of the continuous terrain function f. We refer to [2, 9, 10] for additional information about these discrete terrain models.
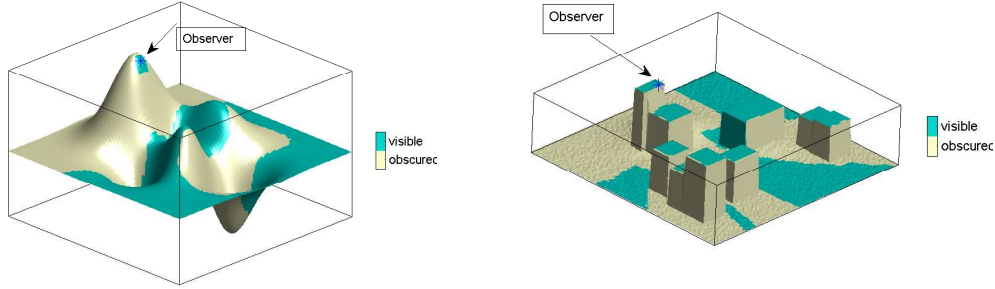

Figure 1: *Examples of a natural (left) and a urban (right) terrain model showing the visible and not visible locations from a viewpoint denoted as the Observer.*

For a fixed $x \in Q$, we say that the point $(y, f(y))$, $y \in Q$, *belongs to the line of sight* of f at x if the interior of the line segment through $(x, f(x))$ and $(y, f(y))$ lies strictly above the surface. In this case, we write that

$$(y, f(y)) \in \mathrm{LOS}_f(x, y),$$

the point $(x, f(x))$ is called a *viewpoint* and y is *visible* from x. It is clear that, if $(y, f(y)) \in \mathrm{LOS}_f(x, y)$, then $(x, f(x)) \in \mathrm{LOS}_f(y, x)$. Hence, two points which belong to the line of sight of each other are mutually visible.

More generally, we can assign an elevation $h \geq 0$ to a viewpoint. Hence, for a fixed elevation $h \geq 0$, we say that a point $(y, f(y))$ belongs to *the line of sight of x with elevation h*, if the interior of the line segment through $(x, f(x) + h)$ and $(y, f(y))$ lies strictly above the surface. In this case, we write:

$$(y, f(y)) \in \text{LOS}_f(x, y; h),$$

and the point $(x, f(x))$ is a *viewpoint with elevation h*.

For each $x \in [0, 1]^2$, we define the *viewshed of f, at the location x with elevation $h \geq 0$*, as the set

$$V_f(x; h) = \{y \in Q : (y, f(y)) \in \text{LOS}_f(x, y; h)\} \subset Q.$$

Hence, $V_f(x; h)$ is the set-valued function consisting of the coordinate points of Q which are visible from the viewpoint $(x, f(x))$ with elevation h (see Figure 1).

It follows from this definition that if $h_1 \leq h_2$ then $V_f(x; h_1) \subset V_f(x; h_2)$.

Equivalent Definitions

For $(x_1, z_1), (x_2, z_2) \in Q \times \mathbb{R}$, let $L((x_1, z_1), (x_2, z_2))$ be the line segment through the points (x_1, z_1) and (x_2, z_2). We define the *viewshed of f at $x \in Q$, with elevation h*, as the set

$$V_f(x; h) = \{y \in Q : \forall z \in \overrightarrow{xy}, \ f(z) < L_z, \ \text{where } (z, L_z) \in L((x, f(x) + h), (y, f(y)))\}.$$

The *Line of Sight of f with elevation h* is

$$\text{LOS}_f(x, y; h) = \begin{cases} L((x, f(x) + h), (y, f(y))), & \text{if } y \in V_f(x; h); \\ \infty, & \text{otherwise.} \end{cases}$$

Other useful visibility structures for a terrain are the *horizons*, describing the locations on Q which belong to the line of sight at the viewpoint x, and block the view of points lying immediately beyond them. Specifically, the *local horizons* of the viewpoint x are the points $p \in Q$ such that $(p, f(p)) \in \text{LOS}_f(x, p; h)$ and there is no $q \in Q$, $q \neq p$, such that $p \in \overrightarrow{xq}$ and all points in $\overrightarrow{pq}$ are visible from x. The *global horizons* of the viewpoint x are the points $p \in Q$ such that $(p, f(p)) \in \text{LOS}_f(x, p; h)$ and for every point $q \in Q$ such that $p \in \overrightarrow{xq}$, q is not visible from x. It is clear that if p is a global horizon point, then it is also a local horizon point.

Several variants of the notion of viewshed are proposed in the literature [4, 5, 6, 7, 11]. Usually, this definition is introduced in the discrete setting and it is denoted as a function of the viewpoint rather than the viewpoint location in Q. Also recall that there are generalized or extended notions of viewsheds [6, 7], where the visibility is restricted within a certain distance from the viewpoint, that is

$$V_f(x, r; h) = \{y \in Q : (y, f(y)) \in \text{LOS}_f(x, y; h) \text{ and } |(y, f(y)) - (x, f(x) + h)| < r\}.$$

Another useful extension is the situation where, for each points $y \in Q$, a richer information than a simple Boolean classification (visible / not visible) is provided. function. Finally, it is also possible to let the elevation h be a function of the viewshed location x, so that $h = h(x)$.

Notice that, in our notation, we make explicit the dependence on the terrain function f, since this functional relation will be specifically considered in the following.

2　Analysis of viewshed function

Our first objective is to examine the properties of the viewshed function $V_f(x; h)$ as a function of the viewpoint location $x \in Q = [0, 1]^2$ and the elevation $h \geq 0$. This will set the groundwork for the analysis of the difference of visibility function which will be introduce in Section 3.

It is clear that, for each $x \in Q$ and each $h \geq 0$, the set $V_f(x; h)$ is non-empty, since it always contains the point $(x, f(x))$. It is also easy to see that, without some regularity assumptions on f, it is virtually impossible to deduce interesting or realistic statements about the viewshed function V_f. In particular, if f is highly irregular, then $V_f(x; h)$ might consist of a single point. For example, let $f(t) = 0$ for rational t and $f(t) = 1$ for irrational t. Then, for each rational x, $V_f(x; h) = \{(x, f(x))\}$ (a singleton set), provided $h < 1$. It is interesting to observe, however, that even if f satisfies some regularity it does not necessarily follow that $V_f(x; h)$ is larger than a singleton. Consider, for example, the terrain model $f(x_1, x_2) = (x_1^2 + x_2^2) \sin\left(\frac{1}{\sqrt{x_1^2 + x_2^2}}\right)$. Notice that f is continuous at the origin, but $V_f((0, 0); 0)$ is the single point $\{(0, 0, 0)\}$. While both examples are not very realistic models of terrains, they are useful to illustrate that, for a more realistic setting, one has to impose some control on the number of discontinuities of f and some regularity on its derivative as well.

Hence, in the following, we will assume that f is a piecewise C^1 function, where the first derivative is bounded. These assumptions are consistent with most terrain models of practical interest, including urban and natural terrains.

For simplicity, in the following we only consider the situation where f is a univariate function; that is $Q = [0, 1]$. The extension to the bivariate case is straightforward for many of the results reported below. The general analysis of the bivariate case will be presented in a successive study. Notice that, since f is defined on a compact domain Q, it follows from our assumptions that f' is uniformly bounded on Q. Also, in the following, we will always assume that the elevation h of the viewpoint satisfies $h > 0$.
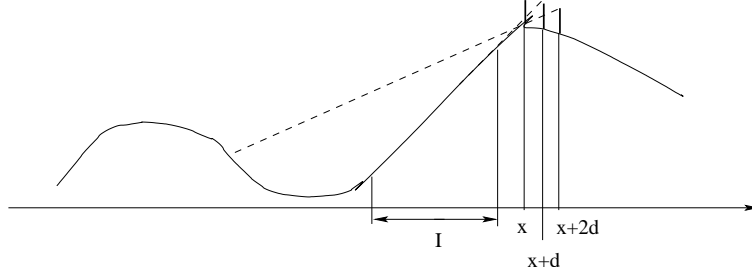
Figure 2: *When the viewpoint location moves from x to $x + d$, the function $V(x)$ decreases abruptly. This is due to the fact that the line of sight has the same slope as the graph of f on a interval I of positive measure. As the viewpoint moves to $x + d$, the whole interval I which was visible becomes invisible.*

2.1 Continuity Properties

It is not difficult to show that, even for a regular terrain f, the set-valued function $V_f(x; h)$ is very sensitive to changes in x and h.

Indeed, let $f \in C^1$ on Q and consider the function $V(x) = \mu\left(V_f(x; h)\right)$. To show that $V(x)$ is not continuous in general, consider the example illustrated in Figure 2. In this case, because the graph of f has constant slope on an interval of positive measure, a small change of the observer location x will change abruptly the size of the viewshed $V_f(x; h)$. It is clear that $V(x)$ is even more sensitive with respect to variations in x if f is discontinuous.

The same terrain f as the one in Figure 2 can be used to show that $V_f(x; h)$ is not a continuous function of h. In fact, it suffices to fix the viewpoint at $x + d$ and vary h so that the interval I is or is not visible.

However, as observed above, we have that:

$$h_1 \leq h_2 \Rightarrow V_f(x; h_1) \subset V_f(x; h_2).$$

Hence the function $U(h) = \mu\left(V_f(x; h)\right)$ is monotonic increasing and lower semicontinuous. Indeed, recall that a function f is *lower semi-continuous* at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) + \epsilon$ for all $x \in U$. Equivalently, this can be expressed as

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

2.2 Set-theoretic Properties of V_f

In this section, we make some observations about the geometry of the set-valued function $V_f(x; h)$. Notice that the assumption $h > 0$ will play an important role in the following.

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The first observation is that $V_f(x; h)$ is a set of nonzero measure, since there is at least one interval (containing the viewpoint x) which is contained in $V_f(x; h)$. Indeed, we have the following observation.

Lemma 2.1. *Let f be piecewise C^1 on Q, with bounded derivative everywhere. Then there exists an interval containing x which is contained in $V_f(x; h)$,*

Proof. Let x be a regular point of f, that is, f is C^1 in a neighborhood of x, and let $|f'(x)| < M$ for each $x \in Q$. It is clear that $x \in V_f(x; h)$. By the continuity of f, for $\epsilon > 0$ small enough, $f(y) < f(x) + h$ for all $y \in (x - \epsilon, x + \epsilon)$. In addition, let $\epsilon > 0$ be such that $\epsilon < h/M$. Hence, for each $y \in (x - \epsilon, x + \epsilon)$, the line from $(x, f(x) + h)$ to $(y, f(y))$ has slope larger than M (indeed, $h/\epsilon > M$). This implies that, for any such $y \in (x - \epsilon, x + \epsilon)$, $(y, f(y) \in \mathrm{LOS}_f(x, y; h)$.

If x is a point of discontinuity of f', a similar argument will hold. Indeed, if x is a jump discontinuity, then there will be an interval $I_1 = [x, x + \epsilon)$ or $I_2 = (x - \epsilon, x]$ such that $(y, f(y) \in \mathrm{LOS}_f(x, y; h)$ for all $y \in I_1$ or $y \in I_2$. $\square$

Notice that, even for x a regular point of f, $V_f(x; h)$ need not be an open or a closed set. In fact, it is easy to find examples where $V_f(x; h)$ contains half-open or half-closed intervals.

The main result of this section states that, under our assumptions on f, then $V_f(x; h)$ must be a finite union of intervals.

Proposition 2.2. *a Let $f \in C^1([a, b])$, where $[a, b]$ is a bounded interval. Then, for each $x \in Q$ and $h > 0$, the set $V_f(x; h)$ is a finite union of intervals.*

In order to prove this result, we first show the following lemma, showing that the line joining a viewpoint to the terrain at a local horizon must be tangent to the terrain at the horizon.

Lemma 2.3. *Let $f \in C^1([a, b])$ and suppose that $p \in (a, b)$ is a local horizon of the viewpoint b with elevation $h > 0$. Then $f'(p) = \frac{f(b)+h-f(p)}{b-p}$.*

Proof. Suppose, by contradiction, that $f'(p) \neq \frac{f(b)+h-f(p)}{b-p}$. If $\frac{f(b)+h-f(p)}{b-p} > f'(p) > 0$, then $(p, f(p) \notin \mathrm{LOS}_f(x, p; h)$, because the line $L((x, f(x) + h), (p, f(p)))$ will either not intersect the graph of f at p or will meet an obstacle before intersecting the graph. The situation is analogous if $f'(p) < \frac{f(b)+h-f(p)}{b-p} < 0$, and also in this case $(p, f(p) \notin \mathrm{LOS}_f(x, p; h)$. On the other hand, if $f'(p) > \frac{f(b)+h-f(p)}{b-p} > 0$, then there is an interval $(p, p+\epsilon)$ which is all contained in $V_f(b; h)$. In fact, there is an $\epsilon > 0$ such that, for all $p^* \in (p, p + \epsilon)$, the tangent line at $(p^*, f(p^*))$ intersects the line $L((b, f(b) + h), (p^*, f(p^*)))$. Hence, p is not a local horizon of b. Similarly, if $0 > f'(p) > \frac{f(b)+h-f(p)}{b-p} > 0$, then there is an interval $(p, p + \epsilon)$ which is all contained in $V_f(b; h)$, and also in this case p is not a local horizon of b. $\square$

Notice that, if $J \in [a, b]$ is an interval where $f \in C^1$ is strictly monotonic and $y_1, y_2 \in J \cap V_f(x; h)$, it does not necessarily follow that $[y_1, y_2] \in V_f(x; h)$. In fact, there may be a

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point $y \in (y_1, y_2)$ where $f'(y) = \frac{f(x)+h-f(y)}{x-y}$ and y is a local horizon. Using this observation, we deduce the following observation.

Lemma 2.4. *Let $f \in C^1([a,b])$. If $y_1, y_2 \in J \cap V_f(x;h)$ and $f'(y) \neq \frac{f(x)+h-f(y)}{x-y}$ for all $y \in (y_1, y_2)$, then $[y_1, y_2] \in V_f(x;h)$.*

Proof. This follows from Lemma 2.3, by observing that no local horizons are possible in the interval (y_1, y_2). □.

Notice that Lemma 2.4 is satisfied if, for example, $y_1, y_2 \in J \cap V_f(x;h)$ and $f''(y) \neq 0$ for all $y \in (y_1, y_2)$.

Proof of Proposition 2.2. If $f \in C^1([a,b])$ and x is a viewpoint with elevation h, one can identify a set S containing all the local horizons of f as follows as the set of the locations p where $f'(p) = \frac{f(x)+h-f(p)}{x-p}$. If this set is discrete then the set $V_f(x;h)$ can be identified as the union of those intervals between distinct local horizons. On the other hand, the set S need not be discrete since it may contain an interval I corresponding to values where f' is constant and it satisfies the equation $f'(p) = \frac{f(x)+h-f(p)}{x-p}$ for all $p \in I \subset S$. However, if this is the case, then the interval I cannot be visible. Hence, the set $V_f(x;h)$ is again determined by the discrete set of points where $f'(p) = \frac{f(x)+h-f(p)}{x-p}$. □

3   Difference of Visibility Distance

Since in both military and civilian applications it is important to compare different terrain models, it is useful to introduce an appropriate notion of distance defined on terrains. Unfortunately, the usual mathematical norms such as the L^p or the Sobolev norms, are not particularly effective to compare terrains on the basis of their visibility properties. In fact, as further illustrated in the examples below, it is easy to produce examples of terrains which are close with respect to an L^p norm, but have completely different visibility properties. In contrast with those norms, we will introduce a new notion of distance which is not defined directly on the space of terrain models and will be useful to precisely compare terrains according to the properties of the associate viewshed functions. Specifically, we define the *Difference of Visibility Distance* between the terrains f and g as:

$$\mathcal{DV}(f,g;h) = \int_Q \mu\left(V_f(x;h) \bigtriangleup V_g(x;h)\right) dx, \tag{1}$$

where $A \bigtriangleup B$ is the symmetric difference of two sets (defined as $(A \cap B^c) \cup (A^c \cap B))$, $h > 0$ is fixed, and μ is the Lebesgue measure. Notice that it is easy to modify the above definition by using the generalized versions of the Visibility Region function mentioned in Section 1.1. However these generalizations will not be considered in this report. Finally, notice that a notion of Difference of Visibility with some similarities to the one given above was recently proposed in [15].

For functions f, g satisfying simple measurability conditions, $\mathcal{DV}(f, g; h)$ is a well-defined function with range on $[0, 1]$ (since we have chosen functions with support on the normalized domain $Q = [0, 1]^2$). In particular, $\mathcal{DV}(f, g; h) = 0$ if f and g are associated with the same viewshed function for all values of the viewpoint.

Notice that $\mathcal{DV}(f, g; h)$ is not properly a distance, in the sense that it does not satisfy the mathematical conditions of a metric; in particular, the triangle inequality does not hold in general.

3.1 Examples

As a first test to motivate and justify this notion of distance, we will compute the Difference of Visibility Distance on a number of simple models of terrains to show that this notion will provide a satisfactory way of assessing the closeness of the viewsheds associated with different terrain models. In the following, we assume that the viewpoint elevation is a fixed constant $0 < h < 1$.
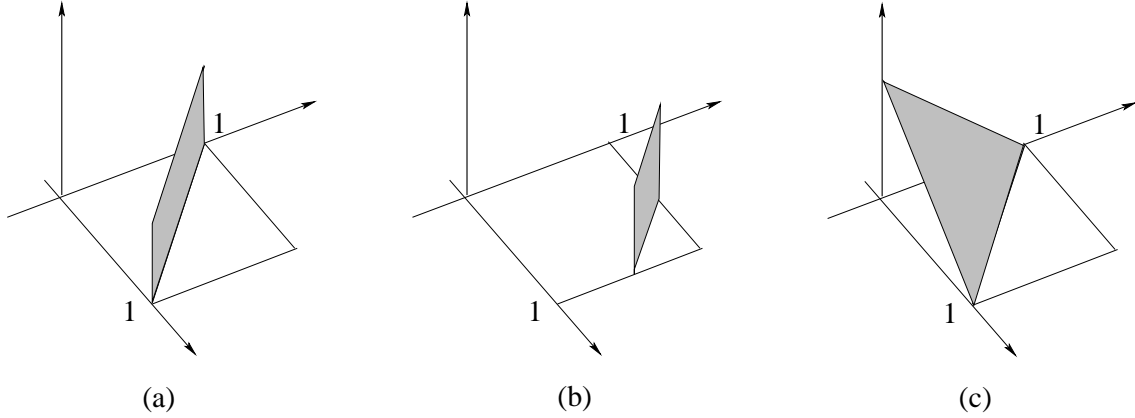

Figure 3: *Examples of map functions: (a) Example 1, (b) Example 2; (c) Example 3.*

Example 1

Let $f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \neq x_2 \\ 1 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 = x_2 \end{cases}$. This is illustrated in Figure 3(a).

Now, if $x = (x_1, x_2)$ is chosen so that $x_1 = x_2$, then $V_f(x; h) = [0, 1]^2$. If $x_1 < x_2$, then $V_f(x; h) = \{(x_1, x_2) : x_1 < x_2\} \bigcap [0, 1]^2$. Finally, if $x_1 > x_2$, then $V_f(x; h) = \{(x_1, x_2) : x_1 > x_2\} \bigcap [0, 1]^2$. Notice that, in the first case, the Lebesgue measure $\mu(V_f(x; h)) = 1$, but in the other two cases $\mu(V_f(x; h)) = 1/2$ (that is, unless the observer is at the ridge $x = y$, only half of the map is visible from any location).

Let $g(x) = 0$ for all $x \in Q$. Notice that the set $V_f(x;h) \triangle V_g(x;h)$ is empty if $x_1 = x_2$, while it is a half-plane if $x_1 \neq x_2$. Since $\mu(V_f(x;h) \triangle V_g(x;h)) = 1/2$ for a.e. $x \in Q$, we conclude that:

$$\mathcal{DV}(f,g;h) = \frac{1}{2}$$

This is consistent with the intuition that g and f are associated with very different viewsheds. In fact, the value of $\mathcal{DV}(f,g;h) = \frac{1}{2}$ can be interpreted as the assessment that *using g as an approximation model for f we would be wrong 50% of the time.*

Notice that, using the traditional L^p norm, with $1 \leq p < \infty$, to measure the distance between f and g, we obtain $\|f - g\|_p = 0$ (while $\|f - g\|_\infty = 1$). This would not be an useful measure for the difference of the visibility sets associated with f and g.

Example 2

Let $f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \neq x_2 + 1/2 \\ 1 & \text{if } (x_1, x_2) \in [0,1]^2 \text{ and } x_1 = x_2 + 1/2 \end{cases}$. This is illustrated in Figure 3(b).

Direct computation shows that $\mu(V_{f_2}(x;h)) = 7/8$ and $\mu(V_{f_2}(x)) = 1/8$, respectively.

Again, letting $g(x) = 0$ for all $x \in Q$ and proceeding as above we find that

$$\mathcal{DV}(f,g;h) = \frac{1}{8}\frac{7}{8} + \frac{7}{8}\frac{1}{8} = \frac{7}{32}$$

Notice that, if the traditional L^p norms are used to measure the distance between f and g, as in Example 1, we obtain $\|f - g\|_p = 0$, for $1 \leq p < \infty$, and $\|f - g\|_\infty = 1$.
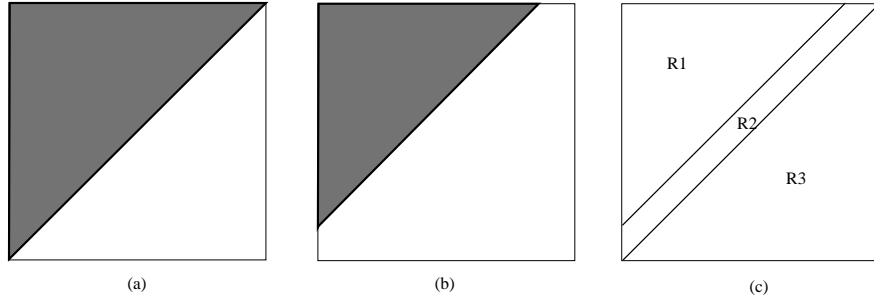

Figure 4: *Example 4: (a) $f_1 = 1$ in the dark set, $f_1 = 0$ in the white set, (b) $f_2 = 1$ in the dark set, $f_2 = 0$ in the white set; (c) $f_1 = f_2$ on the sets $R1$ and $R3$, $f_1 \neq f_2$ on the set $R2$.*

Example 3

Let $f(x_1, x_2) = 1 - x_1 - x_2$, $(x_1, x_2) \in Q$. This is illustrated in Figure 3(c).

For each $x \in Q$, we have that $V_f(x;h) = [0,1]^2$. Hence $\mu(V_f(x;h)) = 1$.

Letting $g(x) = 0$ for all $x \in Q$, we find that $V_f(x; h) \vartriangle V_g(x; h)$ is the empty set, for each $x \in Q$. Hence

$$\mathcal{DV}(f, g; h) = 0.$$

This is the expected result since f and g are associated with the same visibility regions.

Notice that, if the traditional L^p norms are used to measure the distance between f and g, we obtain $\|f - g\|_p \neq 0$. In particular, $\|f - g\|_1 = 1/6$ and $\|f - g\|_\infty = 1$.

Example 4

Let $f_1(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2 \\ 0 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 > x_2 \end{cases}$.

Let f_2 be a perturbed version of f_1: $f_2(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2 - \delta \\ 0 & \text{if } (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 > x_2 - \delta \end{cases}$.

We refer to the notation of Figure 4. We observe that if $x \in R1$, then $V_{f_1}(x; h) \vartriangle V_{f_2}(x; h)$ is empty; if $x \in R2$, then $V_{f_1}(x; h) \vartriangle V_{f_2}(x; h)$ is $R2$; if $x \in R3$, then $V_{f_1}(x; h) \vartriangle V_{f_2}(x; h)$ is empty. Hence, combining these observations, we have that

$$\mathcal{DV}(f_1, f_2) = \mu(R2)\, \mu(R2) = \gamma^2.$$

where $\gamma = \mu(R2)$ is the area of region of perturbation.

3.2   Extensions

The Difference of Visibility $\mathcal{DV}(f, g; h)$ provides a global information about the difference of visibility properties of the terrains f and g. A natural way to introduce a "local" notion for the difference of visibility distance is to consider the quantity:

$$\mathcal{DV}_V(f, g; h) = \int_V \mu\left(V_f(x; h) \vartriangle V_g(x; h)\right) dx, \tag{2}$$

where $V \subset Q$ is the *viewpoint domain*. That is, the viewsheds of the terrains f and g are compared on a subset of Q rather than on the whole domain Q. More generally, it is useful to introduce a weighted version of the Difference of Visibility Distance as

$$\mathcal{DV}_w(f, g; h) = \int_Q \mu\left(V_f(x; h) \vartriangle V_g(x; h)\right) w(x)\, dx,$$

where $w(x)$ is a positive weight function. It is clear that, in the special case where $w(x) = \chi_V(x)$, then we recover the definition (2). In general, by choosing the function w appropriately, it is possible to compare the viewsheds of f and g by weighting in different ways viewpoints located at different regions.
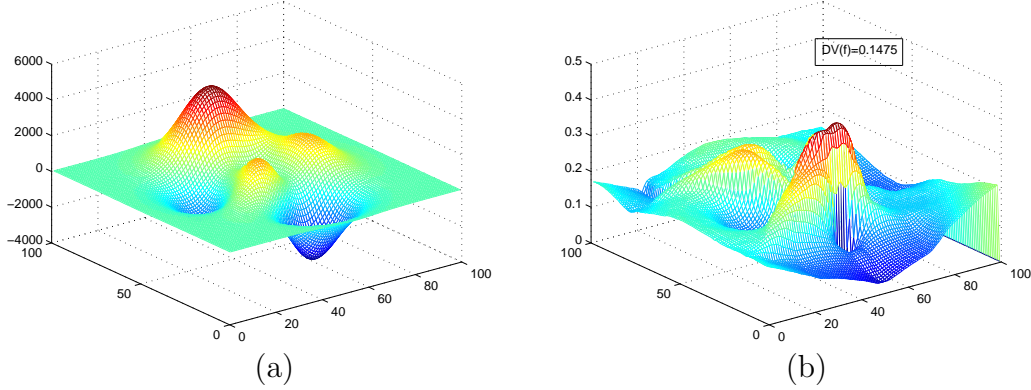
Figure 5: *(a) Map function f, (b) Plot of $\mu(V_f(x))$, as a function of the location x. For this example, a numerical computation gives $\mathcal{DV}(f) = 0.1475$ and, hence, the distance between f and the ground truth g is $\mathcal{DV}(f, g) = 0.8525$.*

As illustrated in some examples in Section 3.1, if $g(x)$ is constant for all $x \in Q$, then the Difference of Visibility Distance $\mathcal{DV}(f, g; h)$ can be interpreted as a measure of the closeness of the terrain f to a terrain which is completely visible. That is, $\mathcal{DV}(f, g; h)$ yields the proportion of the map which is non-visible from a typical location. Indeed, a simple computation shows that

$$\mathcal{DV}(f, g; h) = 1 - \int_{[0,1]^2} \mu\left(V_f(x; h)\right) \, dx$$

and, hence, $\mathcal{DV}(f, g; h) = 0$ if and only if the $V_f(x; h) = 1$ for a.e. $x \in Q$. This shows that the number

$$\mathcal{DV}(f; h) = \mathcal{DV}(f, f; h) = \int_{[0,1]^2} \mu\left(V_f(x)\right) \, dx$$

describes a property intrinsic to the map f.

More precisely, this quantity yields the *expected value of the measure of the visibility region of f for a randomly selected observation point* (assuming uniform distribution on $[0, 1]^2$. This is also *the probability that any two randomly selected locations in the map f are in the Line of Sight of each other* (since the size of the map is normalized to 1).

In particular that, for the map f from Example 1, $\mathcal{DV}(f; h) = 1/2$; for the map f from Example 2, $\mathcal{DV}(f; h) = 25/32$; for the map f from Example 3, $\mathcal{DV}(f; h) = 1$. Each of these numbers gives precisely the probability that a pair of randomly selected points is the line of sight of each other. An example of computation of the number $\mathcal{DV}(f; h)$ for a more general terrain model is given in Figure 5 (the value of $\mathcal{DV}(f; h)$ for this example is computed using a numerical routine which adapts the Matlab functions available in the Mapping Toolbox).

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It is clear that if $\mathcal{DV}(f; h) = 1$ then a single observer is able to have visibility over the whole map f. More generally, if $\mathcal{DV}(f; h)$ is "close" to 1, then most of the map f is visible from a single location. This suggests that $\mathcal{DV}(f; h)$ could be useful as a tool to deal with viewpoint-based placement problems concerning the distribution of observations points on a terrain.

3.3 Analysis of the Difference of Visibility Distance

In this section, we examine the properties of the Difference of Visibility Distance $\mathcal{DV}(f, g; h)$ defined by (1).

We will only consider the univariate case, under the assumption that the terrains f, g are piecewise polynomial functions on $Q = [0, 1]$; that is, they are polynomials except for finitely many points $t_1, \ldots t_N$ where jump discontinuities are allowed. More precisely, for $m = 0, 1, \ldots$, we consider the spline function spaces

$$S_{m,N} = \{f : \ f(t) = \sum_{i=1}^{N} p_i^{(m)}(t) \, \chi_{I_i}(t), t \in Q\},$$

where, for each i, the term $p_i^{(m)}(t)$ is a polynomial of degree m, and $I_i = [t_{i-1}, t_i)$. Notice that we make no regularity assumptions at the knots $\{t_1, \ldots, t_N\}$. In our notation for $S_{m,N}$, the index m is associated with the degree of the polynomial and the index N to the number of knots, corresponding to possible jump discontinuities.

Questions to be investigated include the continuity of the distance of visibility function on these spaces and the construction of "best" approximations with respect to the notion of distance of visibility. For simplicity, in the following we will consider mostly the space $S_{0,N}$; that is, the case of terrain models which are piecewise constant. This will allow us to get a sufficiently clear insight into the properties of the Difference of Visibility Distance.

3.3.1 Continuity.

The first observation is about the continuity of the Difference of Visibility Distance on the space $S_{0,N}$.

Proposition 3.1. *Let $f, g \in S_{0,N}$. Given $\epsilon > 0$, there is a $\delta = \delta(\epsilon, N)$ such that, with probability 1, $\|f - g\|_\infty < \delta$ implies that*

$$\mathcal{DV}(f, g) < \epsilon.$$

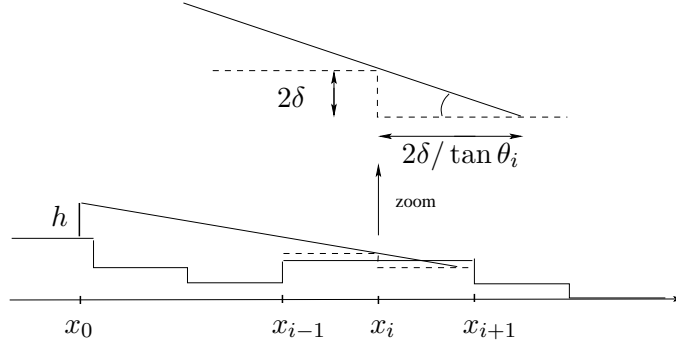To prove this result, we need the following observation.

Figure 6: *A change in f by δ on two consecutive intervals changes the size of the visible region by $\frac{2\delta}{\tan\theta_i}$*

Lemma 3.2. *Let $f, g \in S_{0,N}$, i.e., $f(t) = \sum_{i=1}^{N} c_i \chi_{I_i}(t)$, and suppose that $f(x_0) + h \neq c_i$, for all $i = 1, \ldots, N$. Then, given $\epsilon > 0$, there is a $\delta = \delta(\epsilon, N)$ such that $\|f - g\|_\infty < \delta$ implies that*

$$|\mu(V_f(x_0, h)) - \mu(V_g(x_0, h))| < \epsilon.$$

Proof. Since $f(x_0) + h \neq c_i$ for all i, the slope of each line segment joining $f(x_0) + h$ to $f(x_i)$ satisfies $|\tan\theta_i| > \Delta$ for some $\Delta > 0$. Suppose that $\|f - g\|_\infty < \delta(\epsilon, N) = \frac{\Delta\epsilon}{N}$. It follows (see Figure 6) that, for each pair of consecutive intervals I_i and I_{i+1}, the change of visibility affects at most a subinterval of size $\frac{2\delta}{\tan\theta_i} = \frac{2\epsilon\Delta}{N\tan\theta_i} < \frac{2\epsilon}{N}$. This may happen for at most $N/2$ locations, so that

$$|\mu(V_f(x_0, h)) - \mu(V_g(x_0, h))| < \frac{N}{2}\frac{2\epsilon}{N} = \epsilon. \qquad \square$$

For $x_0 \in Q$ and $h > 0$ being fixed values, let the mapping $\Phi_{x_0,h}$ be defined by

$$\Phi_{x_0,h} : f \to \mu(V_f(x_0, h)). \tag{3}$$

Lemma 3.2 shows that, for each x_0, h, the mapping $\Phi_{x_0,h}$ is continuous on the space $S_{0,N}$, except for a subset of measure zero. Hence we can state that, with probability 1, the mapping $\Phi_{x_0,h}$ is continuous on the space $S_{0,N}$. This also completes the proof of Proposition 3.1.

Similar computations can be used to extend the above results to the space $S_{1,N}$.

3.3.2   Equivalence Properties.

The next group of results are dealing with the issue of establishing a notion of equivalence with respect to the Difference of Visibility Distance. In fact, as observed in Section 3.1, terrains f and g with the same viewshed need not be the same.

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Definition 3.3. *f and g are* **V-equivalent** *if, for all $x \in Q$ and for all $h > 0$, $V_f(x, h) = V_g(x, h)$.*

Proposition 3.4. *Let $f, g \in S_{0,N}$. f and g are V-equivalent if and only if $f = g + c$, where c is a constant.*

Proof.
($\Leftarrow$) It is clear that if the line segment from $f(x) + h$ to $f(y)$, $y \in Q$, meets no obstacle then the same holds for the line segment from $f(x) + c + h$ to $f(y) + c$, for any $c \in \mathbb{R}$.

($\Rightarrow$) Fix $h > 0$ and let $V_f(x, h) = V_g(x, h)$. Suppose that $f(\bar{t}) \neq g(\bar{t})$ for some $\bar{t} \in Q$. Hence we can write

$$f(t) = \sum_{i=1}^{N} c_i(f) \, \chi_{I_i}(t), \quad g(t) = \sum_{i=1}^{N} c_i(g) \, \chi_{I_i}(t),$$

where $c_i(f) \neq c_i(g)$ for some i. Suppose that $c_{i+1}(g) \geq c_i(g)$ and let $x = x_i + \delta$ for some $\delta > 0$. Notice that $V_f(x, h) \cap I_i = V_g(x, h) \cap I_i$ and, thus, it must be that

$$c_{i+1}(g) - c_i(g) = c_{i+1}(f) - c_i(f).$$

A similar argument holds if $c_{i+1}(g) \leq c_i(g)$. Also, the argument can be repeated for each interval I_i. This shows that, for each i, $c_{i+1}(f) = c_{i+1}(g) + c_i(f) - c_i(g)$, so that $f = g + c$ for some constant c. $\square$

Lemma 3.5. *Let $f \in S_{0,N}$ and suppose that $V_f(x, h) = 1$ for all $x \in Q$. Then f is constant.*

Proof. Arguing by contradiction, suppose that f is not constant at t_0. Then f has a jump at t_0. WLOG, let $f(t_0^-) < f(t_0^+)$. It follows that, for $x > t_0^+$, $V_f(x, h) < 1$ since an interval around t_0^- will not be visible from x. This is a contradiction and, hence, f must be constant. $\square$

If $f \in S_{m,N}$, $m > 0$, the argument above does not hold any more. However, we can make the following observation.

Lemma 3.6. *Let $f \in S_{1,N}$ and suppose that $V_f(x, h) = 1$ for all $x \in Q$. Then f is continuous and monotonic.*

Proof. Arguing by contradiction, suppose that f is not constant at t_0. Then f has a jump at t_0. WLOG, let $f(t_0^-) < f(t_0^+)$. It follows that, for $x > t_0^+$, $V_f(x, h) < 1$ since an interval around t_0^- will not be visible from x. This is a contradiction and, hence, f must be continuous. A similar argument shows that f must be monotonic. $\square$

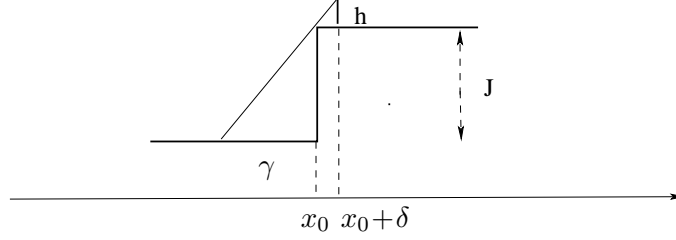We will now show that the mapping Φ defined by (3) is injective on the space $S_{0,N}$.

Figure 7: *A shift of the viewpoint by δ changes the size of the visible region by $\gamma = J\frac{\delta}{h}$*

Proposition 3.7. *Let $f, g \in S_{0,N}$ and suppose that*

$$\mu(V_f(x, h)) = \mu(V_g(x, h)) \quad \text{for all } x \in Q, h > 0.$$

Then f and g are V-equivalent. In fact, $f = g + c$, where c is a constant.

Proof. The first observation is that $\mu(V_f(\cdot, h))$ has a jump whenever f has a jump. Suppose that f has a jump at $x_0 \in Q$, that is,

$$f(x_0^+) - f(x_0^-) = J.$$

WLOG we can assume that $J > 0$. For h sufficiently small, the sampling interval to the left of x_0^- will not be visible from x_0^-, but it will be visible from x_0^+. Also, if the viewpoint is chosen to be x_0^+, then all the region $V_f(x_0^-, h)$ is contained in $V_f(x_0^+, h)$. Hence $\mu(V_f(x_0^+, h)) - \mu(V_f(x_0^-, h)) \geq \frac{1}{N}$ (recall that the size of a sampling interval is $\frac{1}{N}$).

Suppose that x_0 is a regular point. Since $f \in S_{0,N}$, this implies that f is constant near x_0. If the viewpoint x_0 is moved by Δ, then all sightlines emanating from $f(x_0) + h$ will be shifted accordingly. In particular, for each interval of visibility or non-visibility, the size of the visible region is changed by an amount linearly proportional to Δ. This shown that $\mu(V_f(x, h))$ is a continuous function of x, if x is a regular point.

It follows by the observations made above that $\mu(V_f(x, h))$, as a function of x, has a jump discontinuity at x iff f has a jump discontinuity at x. Furthermore, since for each regular point x,

$$\lim_{\Delta \to 0} \frac{\mu(V_f(x + \Delta, h)) - \mu(V_f(x, h))}{\Delta} = c,$$

where c is a constant, then $\mu(V_f(x, h)$ is a piecewise linear function of x.

Finally, let

$$f(x_0^+) - f(x_0^-) = J.$$

We can consider a viewpoint near x_0^+ with elevation h sufficiently small so that "small" changes of the viewpoint location only affects the visibility of the region near x_0. Then, if
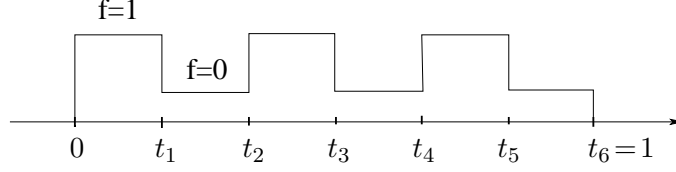
Figure 8: *Example of terrain $f \in S_{0,6}$. There is no approximation of f by $\tilde{f} \in S_{0,3}$ such that $\mathcal{DV}(f, \tilde{f}) < \frac{1}{2}$.*

the viewpoint x_0 is shifted to the right by δ, it follows that $\mu(V_f(x_0+\delta, h)) - \mu(V_f(x_0, h)) = J\frac{\delta}{h}$ (see illustration in Figure 7. This shows that, for δ and h fixed, the visibility change only depends on the jump height J.

Combining all these observations, it follows that if $f \in S_{0,N}$, then $\mu(V_f(x, h))$ completely determine the locations of the regular and irregular points of f, as well as its jump heights. It follows that if $\mu(V_f(x, h)) = \mu(V_g(x, h))$ for all $x \in Q, h > 0$, then f and g can only differ by a constant. $\square$

Open Question. Another natural question is: under which condition is the mapping Φ on $S_{0,N}$ onto? By the proof of Proposition 3.7, we know that if $f \in S_{0,N}$, then $\Phi_{x,h}(f)$ is piecewise linear on $Q \times \mathbb{R}_=$. Conversely, suppose that $\Phi_{x,h}(f)$ is a piecewise linear and bounded function on Q. Does it follow that there exists a function $f \in S_{0,N}$ such that $\Phi_{x,h}(f) = \mu(V_f(x; h))$, where $x \in Q$, $h > 0$?

Observation. Consider the mapping on $S_{0,N}$ given by

$$\Phi_h(f)(x) = \mu(V_f(x; h)),$$

where $h > 0$ is a fixed parameter. Then there is a constant $c > 0$ such that

$$\|\Phi_h(f)\|_{BV} \le c \, \|f\|_\infty,$$

where BV denotes the Bounded variation norm. In fact, the BV norm of $\Psi_h(f)$ is controlled by the L^∞ norm of the (distributional) derivative of $\mu(V_f(x; h))$ and this is controlled by the jump discontinuities of $f \in S_{0,N}$.

3.3.3 Approximations with respect to $\mathcal{DV}$ distance.

One main motivation for the introduction of the Difference of Visibility Distance is the need to compute approximations of terrains which are accurate *with respect to their visibility properties*. Because $\mathcal{DV}$ is not a norm, the approximation of a terrain in terms of the $\mathcal{DV}$ distance is quite different from a projection into a subspace as frequently done in the context of Hilbert spaces.

Consider $f \in S_{0,N}$. We are interested in computing approximations $\tilde{f}$ of f, where $\tilde{f} \in S_{0,M}$, $M < N$, and the approximation is understood in the sense of approximation with
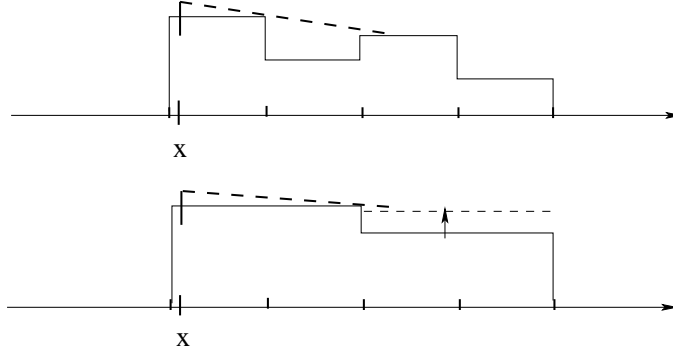
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Figure 9: *A*

respect to the $\mathcal{DV}$ distance. More specifically, our goal is to develop an algorithm such that, starting from $f \in S_{0,N}$, a function $\tilde{f} \in S_{0,N/2}$ is computed such that

$$\mathcal{DV}(f, \tilde{f}) = \min_{g \in S_{0,N/2}} \mathcal{DV}(f, g).$$

This would be useful, for example, in the context of terrain models with variable resolution [10, 12, 16], to compute lower resolution approximations of a terrain which are optimal with respect to their visibility properties.

While we are unable, at the moment, to describe such an algorithm, we have the following important observations.

The first observation is that, given $f \in S_{0,N}$, in general it is not possible to find a $\tilde{f} \in S_{0,N/2}$ such that $\mathcal{DV}(f, \tilde{f}) < \frac{1}{2}$. To show this consider the example of terrain f in Figure 8 which takes only the values 0 or 1. Assume that $h \ll 1$. If the viewpoint is located on one of the 1-valued regions, then $\mu(V_f(x; h)) = 1/2$, while if it is located in one of the 0-valued, then $\mu(V_f(x; h)) = 1/6$. If f is approximated by $\tilde{f}$ where only the even knots are preserved, then the best one can do is to set $\tilde{f} = 1$, so that $\mathcal{DV}(f, \tilde{f}) = 1/2$. No other choice will give a lower values for $\mathcal{DV}$.

The example shows that, in general, we cannot guarantee to produce an approximation such that $\mathcal{DV}(f, \tilde{f}) < 1/2$. In other words:

$$\max_{f \in S_{0,N}} \min_{g \in S_{0,N/2}} \mathcal{DV}(f, g) \geq \frac{1}{2}.$$

The natural question is whether, given any $f \in S_{0,N}$, one can find an approximation $\tilde{f} \in S_{0,N/2}$ such that $\mathcal{DV}(f, \tilde{f}) \leq \frac{1}{2}$. Consider the example of a terrain $f \in S_{0,4}$ illustrated in Figure 9, with a fixed viewpoint x. It is clear that taking the average value of f over consecutive subintervals will provide no control on the Difference of Visibility Distance. As an alternative strategy, one could assign the highest value of two consecutive subintervals, as indicated in Figure 9 (bottom). At this point, in order to ensure that visible regions are still visible, one could adjust the height of the new partition intervals, to compensate for occlusions due to the new sampling.

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4   Summary of Results and Future Work

The main contribution of this report is to introduce a novel notion of difference of visibility distance which overcomes the limitation of traditional mathematical norms and is consistent with human-based goals as understood in application from surveillance, reconnaissance, navigation, asset emplacement, robotics, and situational awareness. This method is especially designed to compare terrains from the point of view of their visibility properties and the examples described in this report clearly demonstrate the advantages of the new Difference of Visibility Distance with respect to the traditional mathematical norms which are frequently used to compare terrains. Furthermore, our new approach opens up a completely new perspective into the study of the visibility problem, including new notions of terrain approximation, representation and equivalence which are derived from the new definition of distance. In addition, our framework leads to an intrinsic characterization of terrains from the point of view of their viewshed. Within the limited scope of this project, only simplified models of terrains have been considered, mostly in the univariate case. In this setting, we have established conditions for terrains to be equivalent with respect to their visibility properties; we have established conditions for continuity with respect to the Difference of Visibility Distance; we have investigated a number of properties dealing with the approximation of terrains with respect to this distance. Indeed, the analysis of the Difference of Visibility Distance presented in this report provides only an exploratory investigation about the properties of this new notion of distance and its applications to the study of visibility problems. It is clear that more advanced analytical tools will be needed to deal with more general problems of approximation of terrains using this framework. Also, a new generation of numerical algorithms will be needed to incorporate these ideas into effective methods for visibility computations.

Future work includes the extension of a number of one-dimensional theoretical results described to the two-dimensional setting. This is expected to be straightforward in most cases. More importantly, a deeper investigation will be needed about the notion of approximation with respect to $\mathcal{DV}$ distance, in order to understand the ultimate potential of an approach based on this distance. In particular, it will be very important to examine the impact of the $\mathcal{DV}$ distance into existing strategies for visibility analysis and computation based on hierarchical or multiresolution terrain models. Using these methods, the terrain is modeled with variable local resolution, in order to reduce the computational effort by focusing selectively on regions of special interest. However, the effectiveness of this strategy depends heavily on the ability to accurately approximate terrains at coarser resolution. This is precisely the situation where the newly introduced $\mathcal{DV}$ distance will be able to provide more accurate and effective strategies for terrain approximations.

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